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THE CONVERGENCE OF SEQUENCES OF 'FIXED POINTS IN 2-METRIC SPACES

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Abstract: The mathematician Rhoades B.E[1] studied the aspects of fixed point theory in 2-metric space and give theorems on the convergence of sequences of fixed points in 2-metric spaces. In this paper we discuss on some results of fixed point theorems of mappings considered by Rhoades B.E[1].

Key words: 2-metric space, Cauchy Sequences, Complete metric spaces.

INTRODUCTION:

The notion of 2-metric space was introduced by Gahler S.[2]. More recently Iseki K.[3], Iseki et al[4] and Rhoades B.E[1] studied the aspects of fixed point theory in 2-metric space and give the theorem on the convergence of sequence of fixed points in 2-metric spaces.

DEFINITION 1.1 :

A 2-mertic space is a space X in which for each triple of points a, b, c there exists a real-valued function $\rho(a, b, c)$ such that following hold:

(i) To each pair of points a, b with $a \neq b$ in X there exists a point $c \in X$ such that $\rho(a, b, c) \neq 0$;
(ii) $\rho(a, b, c) = 0$ when at least two of the points are equal;
(iii) $\rho(a, b, c) = \rho(b, c, a) = p(a, c, b)$;
(iv) $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$.

It is easily seen that p is non-negative.

DEFINITION 1.2: A sequence $\{x_n\}$ in X is 'convergent' and $x \in X$ is the limit of the sequence if $\lim_{n\to\infty} \rho(x_n, x, a) = 0$ for all $a \in X$.

DEFINITION 1.3:A sequence $\{x_n\}$ in X is a 2-metric space X is called 'Cauchy sequence' if $\lim_{n,m\to\infty} \rho(x_n, x_m, a) = 0$ for all $a \in X$. If every Cauchy sequence is convergent, X is called a 'complete 2-metric space".

DEFINITION 1.4 :

A 2-metric which is continuous in all its three arguments is called 'continuous'.

THEOREM A: –

Let X be a complete 2-metric space, $T: X \to X$ satisfying: "there exists a q, $0 \leq q < 1$ such that for each $x, y, a \in X$ (*) $\rho(Tx, Ty, a) \leq q \max\{\rho(x, y, a), \rho(x, Tx, a), \rho(y, Ty, a), \rho(x, Ty, a), \rho(y, Tx, a)\}$, then T possesses a unique fixed point z and lim $T^n x_o = z$ for each $x_0 \in X_n$

THEOREM B: –

Let X be a complete 2-metric space with ρ continuous, $\{T_n\}$ as sequence of mapping $T_n : X \to X$ satisfying (*) for each n and same q, such that T_n, tends pointwise to a function T. Then T has a unique fixed point z and $z_n \to z$, where the z_n are the unique fixed points of T_n.

It may be remarked that Theorem A and Theorem B are 2-metrie space versions of fixed point theorems due to Ciric L.J[(5,6)] Also the contractive definition (*) is one of the most general definitions possible for 2-metric spaces.

THEOREM 1.5 :

Let (X, ρ_0) be a 2-metric-space. ρ_n is sequence of 2-metrcs on X converging uniformly to ρ_0. Let $\{T_n\}$ be a sequence of self-mappings on X converging ρ_0-pointwise to a map T with fixed point z and let T_n having fixed points z_n satisfying (*) with respect to ρ_n, for each n and same q. Then $z_n \to z$.

PROOF :

The uniform convergence of ρ_n to ρ_0 implies that for any z > 0 and x, y, a $\in$ z one gets

$$|\rho_n (x, y, a) - \rho_0 (x, y, a)| < \left(\frac{1-q}{3+q}\right)$$

Also by the point wise convergence of T_n to T with respect to the 2-metric ρ_0 yields that for all a $\in$ X

$$\rho_0 (Tz, T_n z, a) < \left(\frac{1-q}{3+q}\right)$$

whenever $n \geq N$ for some natural number N. Now for $n \geq N$, we have

$$\begin{aligned}\rho_0 (z, z_n, a) &= \rho_0 (Tz, T_n z_n, a)\\ &\leq \rho_0 (Tz, T_n z_n, T_n z) + \rho_0 (Tz, T_n z, a) + \rho_0 (T_n z, T_n z_n, a)\\ &\leq \rho_n (Tz, T_n z_n, T_n z) + \rho_0 (Tz, T_n z, a) + \rho_n (T_n z, T_n z_n, a)\\ &\quad + \left(\frac{1-q}{3+q}\right)\in + \left(\frac{1-q}{3+q}\right)\in\end{aligned}$$

From Theorem A&B ,we conclude that,

$$\rho_n (T_n z, T_n z_n, a) \leq q \max \{\rho_n (z, z_n, a) + \rho_n (z, T_n z, a)\} \qquad \ldots(1.1)$$

Hence

$$\rho_n (Tz, T_n z_n, T_n z) = \rho_n (T_n z_n, T_n z, Tz) = 0$$

Considering two cases of (1.1), we get

$$\begin{aligned}\rho_n (T_n z, T_n z_n, a) &\leq q\, \rho_n (z, z_n, a)\\ &\leq q \left[\rho_0(z, z_n, a) + \left(\frac{1-q}{3+q}\right)\in\right]\end{aligned}$$

Here second alternative in (1.1) is not admissible. "

Thus

$$\begin{aligned}\rho_0 (z, z_n, a) &\leq \rho_0 (Tz, T_n z, a)\\ &\quad + \left[q\rho_0(z, z_n, a) + q\left(\frac{1-q}{3+q}\right)\in\right]\\ &\quad + \left(\frac{1-q}{3+q}\right)\in + \left(\frac{1-q}{3+q}\right)\in\end{aligned}$$

$$\rho_0 (z, z_n, a) \leq \frac{\rho_0(Tz, T_n z, a)}{(1-q)} + \left(\frac{2+q}{1-q}\right)\left(\frac{1-q}{3+q}\right)\in < \in$$

which implies the convergence of z_n to z.

REMARKS1.6:

(1) Theorem B is a particular case of our Theorem (1.5) when $\rho_n = \rho_0$ for each n.

(2) The existence of the unique fixed point z of T can be proved under the extra assumption that ρ_n is continuous for each n and via the use of condition (*).

THEOREM 1.7: Let $\{T_n\}$ and $\{S_n\}$ be sequences of mappings of a 2-mctric space X with ρ continuous and for each n satisfying $\rho_0 (T_n x, S_n y, a) \leq q \max \{\rho (x, T_n x, a), \rho (y, S_n y, a), \rho (x, y, a), \rho (x, S_n y, a), \rho (y, T_n x, a)\}$ for all x, y, a $\in$ X, q a fixed constant satisfying $0 \leq q < 1$. If $\{T_n\}$ and $\{S_n\}$ converge pointwise to mappings T and S on X respectively, the following three statements are equivalent:

(i) T has a fixed point,
(ii) S has a fixed point,
(iii) $\{x_n\}$ is a convergent sequence where $\{x_n\}$ is a common fixed point of T_n and S_n for each n.

PROOF :

We shall prove the theorem by the use of following lemmas.

LEMMA 1.8:
If x_n and x are fixed points of S_n and T respectively, and if $\{T_n\}$ converges pointwise to T, then $\{x_n\}$ converges to x.

PROOF:
For each natural number n, consider

$\rho(x, x_n, a) = \rho(Tx, S_n x_n, a)$

$\leq \rho(Tx, S_n x_n, T_n x) + \rho(Tx, T_n x, a) + \rho(T_n x, S_n x_n, a)$

From the given condition

$\rho(T_n x, S_n x_n, a) \leq q \max\{\rho(x, x_n, a), \rho(x, T_n x, a)\}$

so that

$\rho(Tx, S_n x_n, T_n x) = (T_n x, S_n x_n, Tx) = 0$

Thus

$\rho(x, x_n, a) \leq \rho(Tx, T_n x, a) + q \max\{\rho(x, x_n, a), \rho(Tx, T_n x, a)\}$

which gives

$\rho(x, x_n, a) \leq \rho(Tx, T_n x, a)(1 - q)^{-1} \to 0$ as $n \to \infty$.

LEMMA 1.9:
If x_n is a fixed point of S_n for each n and $x_n \to x$, x is a fixed point or T where T is point wise limit of T_n.

PROOF :As in the proof of Lemma (1.8) $\rho(T_n x, S_n x_n, a) \leq q \max\{\rho(x, x_n, a), \rho(x, T_n x, a), \rho(x_n, T_n x, a)\}$ Using continuity of ρ and letting $n \to \infty$, one has $\rho(Tx, x, a) \leq q\,\rho(x, Tx, a)$, This gives $\rho(Tx, x, a) = 0$ for all $a \in X$. Hence $Tx = x$, otherwise $P(Tx, x, a) \neq 0$ for some a.

PROOF OF THEOREM 1.7
Since (i) implies (iii) by Lemma (1.8) and (iii) implies (i) by Lemma (1.9) equivalence of (i) and (iii) is established. In a similar way, we can prove the equivalence of (ii) and (iii). Hence the proof.

REFERENCES.

[1] Rhoades B.E: Contractive type mapping on 2-metric space, Math. Nachr, 151-155 (1979).
[2] Gahler S. :Uber-2 Banach Raume Math.Nachr.42(1969),335-347.
[3] Iseki K.: Common fixed point of mapping, Tamkang J.Math.62(2)(1975),157-159.
[4] Iseki K:Fixed point theorems for densifying mappings,Nauta Math,9(1976),50-53
[5] Ciric,L.J :Generalized contractions and fixed point theorem .Publ .Inst .Math((Beograd) (N.S) 12(26) (1971),19-26.
[6] Ciric,L.J :On some maps with non-unique fixed point .Publ .Inst .Math((Beograd) (N.S) 17(31) (1974),52-58.

